

Banach-like metrics and metrics of compact sets.

A. Duci, A. C. Mennucci

April 26, 2009

Abstract

We present and study a family of metrics on the space of compact subsets of \mathbb{R}^N (that we call “shapes”). These metrics are “geometric”, that is, they are independent of rotation and translation; and these metrics enjoy many interesting properties, as, for example, the existence of minimal geodesics. We view our space of shapes as a subset of Banach (or Hilbert) manifolds: so we can define a “tangent manifold” to shapes, and (in a very weak form) talk of a “Riemannian Geometry” of shapes. Some of the metrics that we propose are topologically equivalent to the Hausdorff metric; but at the same time, they are more “regular”, since we can hope for a local uniqueness of minimal geodesics.

We also study properties of the metrics obtained by isometrically identifying a generic metric space with a subset of a Banach space to obtain a rigidity result.

Keywords. Shape, Shape Optimization, Shape Analysis.

Acknowledgments. A. Duci was supported by the EU 6th Framework Programme Grant MRTN - CT - 2005 - 019481.

1 Introduction

A wide interest for the study of *shape spaces* arose in recent years, in particular inside the Computer Vision community.

There are two different (but interconnected) fields of applications for a good Shape Space in Computer Vision:

Shape Optimization where we want to find the shape that best satisfies a design goal; a topic interest in Engineering at large;

Shape Analysis where we study a family of Shapes for purposes of statistics, (automatic) cataloging, probabilistic modeling, among others, and possibly to create an a-priori model for a better Shape Optimization.

To achieve the above, some structure is clearly needed on the Shape Space, so that our goals can be studied and the problem can be solved.

Remark 1.1. Note that, for the purpose of Shape Optimization, shapes are usually intended “up to rotation, translation and scaling”; for this reason, when we wish to distinguish between the two, we will call a space for Shape Optimization a “prespace”.

1.1 Shape spaces

In general the “Shape Space” \mathcal{I} will be a suitable choice of subsets of \mathbb{R}^N .

A common way to model shapes is by **representation/embedding**:

- we **represent** the shape A by a function u_A
- and then we **embed** this representation in a space E , so that we can operate on the shapes A by operating on the representations u_A ;

for example, if E is a Banach space with norm $\|\cdot\|$, we can define a *distance of shapes* by $d(A, B) \stackrel{\text{def}}{=} \|u_A - u_B\|$.

Most often, this representation/embedding scheme does not directly provide a Shape Space satisfying all desired properties. In particular, in many cases it happens that the representation is “redundant”, that is, the same shape has many different possible representations. An appropriate **quotient** is then introduced.

There are many examples of the *representation/embedding/quotient* scheme in the literature; for the case of generic subsets of \mathbb{R}^N ,

- a standard representation is obtained by associating a closed subset A to the **distance function**

$$u_A(x) \stackrel{\text{def}}{=} \inf_{y \in A} |x - y| \quad (1)$$

or the **signed distance function**

$$b_A(x) \stackrel{\text{def}}{=} u_A(x) - u_{\mathbb{R}^N \setminus A}(x) \quad (2)$$

We can then define a *topology of shapes* by deciding that $A_n \rightarrow A$ when $u_{A_n} \rightarrow u_A$ uniformly on compact sets. This convergence coincides with the Kuratowski topology of closed sets.

We can also operate “linearly” on shapes by operating on u_A or b_A : so we can define *shape averages* and *shape principal component analysis*. Note that in general a linear combination of (signed) distance functions will not be a (signed) distance function: so any linear operation must be followed by an *ad hoc* correction. For example, given two shapes A_0, A_1 , we can define an interpolation A_t for $t \in [0, 1]$ by setting $A_t = \{x \mid tb_{A_1}(x) + (1 - t)b_{A_0}(x) \leq 0\}$.

This Shape Space is not independent of the position: when it is used for shape analysis, a *registration* of the shapes to a common pose is often performed (but, see also sec. 2.1.1).

- A. Duci *et al* (see [7, 8]) represent a closed planar contour as the zero level of a harmonic function. This novel representation for contours is explicitly designed to possess a linear structure, which greatly simplifies linear operations such as averaging, principal component analysis or differentiation in the space of shapes.
- Trouvé–Younes *et al* (see [9], [27] and references therein) modeled the motion of shapes by studying a left invariant Riemannian metric on the diffeomorphisms of the space \mathbb{R}^n ; to recover a true metric of shapes, a quotient is then added.

But the *representation/embedding/quotient* scheme is also found when dealing with spaces of curves:

- In the work of Mio, Srivastava *et al.* [20, 19, 11], smooth planar closed curves $c : S^1 \rightarrow \mathbb{R}^2$ of length 2π are parametrized by arclength and represented by the angle function $\alpha[0, 2\pi] \rightarrow \mathbb{R}$ such that

$$\dot{c}(s) = (\cos(\alpha(s)), \sin(\alpha(s)))$$

then the angle function is embedded in a suitable subspace N of $L^2(0, 2\pi)$ or $W^{1,2}(0, 2\pi)$. Since the goal is to obtain a Shape Space representation for Shape Analysis purposes, a quotient is then introduced on N .

- Another representation of planar curves for Shape Analysis is found in Younes [31]. In this case the angle function is considered $\text{mod}(\pi)$. This representation is both simple and very powerful at the same time. Indeed, it is possible to prove that geodesics do exist and to explicitly show examples of geodesics.
- Metrics of “geometric” curves (that is, curves up to the choice of parametrization) have been studied by Michor–Mumford [18, 17, 16] and Yezzi–Mennucci [30, 28, 29]; more recently, Yezzi–Mennucci–Sundaramoorthi [26, 25, 24, 23, 15, 21, 22] have studied Sobolev–like metrics of curves and shown many good properties for applications to Shape Optimization; similar results have also been shown independently by Charpiat *et al* [5].

Remark 1.2. In this case, shapes are modeled as immersed parametric curves $c : S^1 \rightarrow \mathbb{R}^N$, for the sake of mathematical analysis; a quotient w.r.t the group of possible reparametrizations of the curve c (that coincides with the group of diffeomorphisms $\text{Diff}(S^1)$) is applied afterward to all the mathematical structures that are defined (such as the manifold of curves, the Riemannian metric, the induced distance, etc.).

1.2 Goals

We remarked that, in Shape Analysis, shapes are usually considered “up to rotation, translation and scaling”, but even in Shape Optimization, to a certain degree, our theory should be independent of rotation and translation: that is, whatever we do with shapes should not depend on “where in the plane” we do it.

In the rest of the paper we will denote by \mathcal{I} the family of the nonempty compact sets in \mathbb{R}^N , and we will build many examples of metrics d on \mathcal{I} . We will always require these metrics to be **euclidean invariant**. If A is an euclidean transformation of the space (a rigid transformation), then

$$d(A\Omega_1, A\Omega_2) = d(\Omega_1, \Omega_2) .$$

What other properties may be interesting for applications?

As mentioned before, a goal of Shape Optimization is to define *shape metrics*, *shape averages*, *shape principal component analysis*, *shape probabilities*...

For example, if we represent shapes $A_j, j = 1 \dots n$ by their *signed distance function* b_{A_j} , then we may define *Signed Distance Level Set Averaging*

$$\bar{A} = \left\{ x \mid f(x) = 0 \right\}, \text{ where } f(x) = \frac{1}{N} \sum_{n=1}^N b_{A_n}(x) \quad (3)$$

A benefit of this definition is that it is easily computable; a defect is that, if the shapes are far away, then \bar{A} will be empty. Another defect is that this definition is quite *ad hoc*:

it is not coupled with any other structure that we may wish to add to the Shape Space, such as a metric d . We may then look at the problem in the other direction.

Considering a generic metric space (M, d) , define the *Distance Based Averaging*¹ of any given collection $a_1 \dots a_n \in M$, as a minimum point \bar{a} of the sum of its squared distances:

$$\bar{a} = \arg \min_a \sum_{j=1}^n d(a, a_j)^2 \quad (4)$$

Supposing now that the Shape Space \mathcal{I} is given a metric d , we can use the abstract definition above to define *shape averages*; this definition has many advantages. Namely

- it comes from a minimality criterion, so it is “*optimal*” in a certain sense (contrary to the definition (3)).
- If the distance is invariant w.r.t. a group action, then the *shape average* is as well (see sec. 2.1.1). For example, in the case of *geometric curves*, where the distance is independent of parametrization, then the *shape average* will be independent of the parametrization of $a_1 \dots a_n$.
- It coincides with the arithmetic mean in Euclidean spaces; more in general, when \mathcal{I} is a smooth submanifold of a Banach space and $a_1 \dots a_n$ are near enough, then \bar{a} is an approximation of the arithmetic mean.

In particular, the average of two shapes A_1, A_2 is the *midpoint*, that is a shape A such that

$$d(A_1, A) = d(A_2, A) = \frac{1}{2}d(A_1, A_2)$$

We are then, however, bound by this result (whose base idea goes back to Menger – see sec. §4.i.1 in [14] for more details)

Theorem 1.3. *Supposing that d is complete and intrinsic (see sec. 2.1), then the following facts are equivalent:*

- *for any two shapes A_1, A_2 there is a midpoint;*
- *for any two shapes A_1, A_2 there is a minimal geodesic connecting them.*

For this reason, we end up studying whether the Shape Space admits minimal geodesics (in theorem 3.18).

1.2.1 Tradeoff

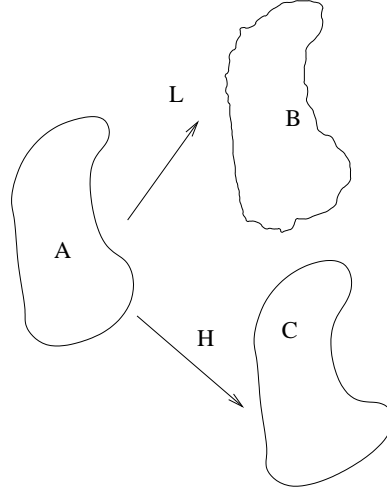
Unfortunately a tradeoff (that is well known in the Calculus of Variations) arises;

- on one hand, a Shape Space that is useful for Shape Optimization should possibly be equipped with a topology that makes functionals “regular”, so that suitable minimization methods can be used; to this end, the topology should have many open sets.
- On the other hand, to prove existence of average points and of geodesics (that are useful in Shape analysis), it is sufficient that certain bounded sets be compact (cf. 2.2, 2.3, 2.4): to this end, the topology should have few open sets.

¹also known as *Karcher mean*, but it is also sometimes attributed to Fréchet, in 1948

We can exemplify this as follows.

- As we mentioned before, it was shown in [25, 24] that flows for Active Contour methods that use a Sobolev metric are more robust to noise and converge faster than standard flows. To explain the rationale, suppose H, L are two metrics, H being stronger than L . When evolving the shape A , the L -flow will move to-



wards a shape B with small scale deformations (such as those induced by noise), since B is nearer to A in the L -induced distance; whereas the H -flow will move towards the shape C with large scale deformations, since C is nearer to A w.r.t its related distance.

- Suppose now though that a dataset contains a template shape A ; an algorithm is given a version B of A that was corrupted by noise, and different shape C , and it has to decide what is the best match to A . In this case, the weaker metric L would associate A to the correct shape B , whereas an algorithm using the metric H would fail to associate A to B .

For all those reasons, it is quite difficult to find a Shape Space that is suited both for Shape Analysis and for Optimization.

1.3 Plan of the paper

The plan of the paper is as follows: we foremost provide base definitions, and we propose some results in the theory of metric spaces, in particular when they are isometrically embedded in Banach spaces. Considering the space \mathcal{I} of compact sets we review the well-known Hausdorff distance, and its properties; we successively propose a class of metric spaces that are similar to the Hausdorff distance, while at the same time enjoying some extra properties that may be useful in applications.

2 Metric spaces and embeddings in Banach spaces

2.1 Metric spaces

We recall some basilar definitions and results in the abstract theory of metric spaces. Suppose that (M, d) is a metric space. We induce from d the length $\text{len}^d \gamma$ of a continuous curve $\gamma : [\alpha, \beta] \rightarrow M$, by using the total variation

$$\text{len}^d \gamma \stackrel{\text{def}}{=} \sup_T \sum_{i=1}^n d(\gamma(t_{i-1}), \gamma(t_i)), \quad (5)$$

where the sup is carried out over all finite subsets $T = \{t_0, \dots, t_n\}$ of $[\alpha, \beta]$ and $t_0 \leq \dots \leq t_n$.

We define the **induced geodesic distance** d^g by

$$d^g(x, y) \stackrel{\text{def}}{=} \inf_{\gamma} \text{len}^{d^1} \gamma, \quad (6)$$

where the inf is taken in the class of all continuous curves γ connecting x to y . If the inf is a minimum, the curve providing the minimum is called a *geodesic*. Note that it may be the case that $d^g(x, y) = \infty$ for some choices of x, y . Note also that $d^g \geq d$.

We may consider the metric space (M, d^g) ; but the topology of (M, d) and (M, d^g) may be quite different, as we see in this example:

Example 2.1. Let us consider the subset $M \stackrel{\text{def}}{=} \psi(E)$ of \mathbb{R}^2 , where

$$E \stackrel{\text{def}}{=} [0, 1] \times \left(\{0\} \cup \{1/n \mid n > 0\} \right)$$

and

$$\psi(\rho, \theta) \stackrel{\text{def}}{=} \rho(\cos(\theta), \sin(\theta)).$$

(See fig. 1). We associate to M the distance d induced by the usual distance of \mathbb{R}^2 onto M and d^g for the geodesic distance. Then we have that (M, d) is obviously compact whereas (M, d^g) is not: indeed $x_n \stackrel{\text{def}}{=} \psi(1, 1/n)$ does not admit a converging subsequence, since $d^g(x_n, y_m) = 2$ for all $n \neq m$.



Figure 1: example 2.1

When $d = d^g$, we will say that the metric space is **path-metric**, or that d is **intrinsic**.

Note that the length len^{d^g} defined by d^g coincides with len^d , and then $d^g = (d^g)^g$: d^g is always intrinsic.

We will use the following proposition:

Proposition 2.2. *if for a choice of $\rho > 0$*

$$\mathbb{D}^g(x, \rho) \stackrel{\text{def}}{=} \{x \mid d^g(x, y) \leq \rho\} \quad (7)$$

is compact in the (M, d) topology, then x and any $y \in \mathbb{D}^g(x, \rho)$ may be connected by a geodesic.

The proof is simply obtained by the direct method in the Calculus of Variations (see Thm. 4.24 in [14]).

We also state these simple propositions.

Proposition 2.3. *If for a $x \in M$ for all choices of $\rho > 0$*

$$\mathbb{D}(x, \rho) \stackrel{\text{def}}{=} \{x \mid d(x, y) \leq \rho\}$$

is compact then (M, d) is complete.

Proposition 2.4. *Suppose that $a_1 \dots a_n \in M$ are given; a sufficient condition for the existence of the Geodesic Distance Based Averaging \bar{a} of $a_1 \dots a_n$*

$$\bar{a} = \operatorname{argmin}_a \tau(a), \text{ where } \tau(a) \stackrel{\text{def}}{=} \sum_{j=1}^n d^g(a, a_j)^2 \quad (8)$$

is that, defining

$$\rho^* = \min_{i=1, \dots, n} \tau(a_i)$$

we have that $\rho^ < \infty$ and that $\mathbb{D}^g(a_1, 2\sqrt{\rho^*} + \varepsilon)$ is compact in the (M, d) topology, for $\varepsilon > 0$ small.*

Proof. Note first that the infimum of $\tau(a)$ is finite, since it does not exceed ρ^* . Recall that

$$d^g(a, a_j) = \inf_{\gamma_j} l_j$$

where l_j is the length of a Lipschitz curve γ_j connecting a, a_j . So we can rewrite the problem (8) as

$$\inf_{\gamma_1 \dots \gamma_n} \theta(\gamma_1 \dots \gamma_n), \quad \text{where } \theta(\gamma_1 \dots \gamma_n) \stackrel{\text{def}}{=} \sum_{j=1}^n (l_j)^2$$

where the infimum is computed on all choices of Lipschitz curves $\gamma_1 \dots \gamma_n$ of length $l_1 \dots l_n$ connecting a_i to a common point $x \in M$; for simplicity we represent them as $\gamma_i : [0, l_i] \rightarrow M$ parametrized by arc parameter. By triangular inequality

$$d^g(a_i, \gamma_j(t)) \leq d^g(a_i, x) + d^g(x, \gamma_j(t)) \leq l_i + l_j$$

Let then $\gamma_{i,k}$ be a sequence of choices that converges to the infimum:

$$\theta(\gamma_{1,k} \dots \gamma_{n,k}) \rightarrow_k \inf_{\gamma_1 \dots \gamma_n} \theta(\gamma_1 \dots \gamma_n)$$

so for large k ,

$$\theta(\gamma_{1,k} \dots \gamma_{n,k}) \leq \rho^* + \varepsilon$$

but then in particular $l_{i,k} \leq \sqrt{\rho^* + \varepsilon}$ hence

$$d^g(a_i, \gamma_j(t)) \leq 2\sqrt{\rho^* + \varepsilon}$$

so all the curves are contained in a compact set. By Ascoli–Arzelà theorem, we can then extract a uniformly convergent subsequence, and use the fact that the length is lower semi continuous. \square

A similarly proposition can be stated for d :

Proposition 2.5. *Suppose that $a_1 \dots a_n \in M$ are given; let*

$$\rho^* = \min_i \sum_{j=1}^n d(a_i, a_j)^2 \quad (9)$$

and i^ the index that achieves the above minimum: suppose that $\mathbb{D}(a_{i^*}, \sqrt{\rho^*} + \varepsilon)$ is compact for $\varepsilon > 0$ small: then there exists a point \bar{a} that is the Distance Based Averaging of $a_1 \dots a_n$, as defined in (4).*

2.1.1 Distances, quotients and groups

Let $d_M(x, y)$ be a distance on a space M , and G a group acting on M ; a distance d_B may be defined on $B = M/G$ by

$$d_B([x], [y]) = \inf_{x \in [x], y \in [y]} d_M(x, y) = \inf_{g, h \in G} d_M(gx, hy)$$

that is the lowest distance between two orbits; we write $d_B(x, y)$ for simplicity.

If d_M is **invariant w.r.t.** G , i.e.

$$d_M(gx, gy) = d_M(x, y) \quad \forall g \in G$$

then

$$d_B(x, y) = \inf_{g \in G} d_M(gx, y) \quad (10)$$

It is easy to see that d_B satisfies the triangular inequality; but it may be the case that $d_B(x, y) = 0$ even when $x \neq y$. We state a simple sufficient condition

Lemma 2.6. *If the orbits are compact, then d_B is a distance.*

When studying metrics d on a Shape Space \mathcal{I} , the quotient is particularly useful in at least two cases:

- when we want to pass from a *preshape space*² to a *shape space*: in this case, G is the Euclidean group of rotations and translation (and sometimes of scaling);
- when the representation is redundant: for example, in remark 1.2 we would set $G = \text{Diff}(S^1)$.

2.2 Embeddings in Banach spaces

In most of what follows, we will be able to identify M (using an isometry i) with a subset N of a Banach space E . We remark that an isometry is a map i such that $d(x, y) = \|i(x) - i(y)\|$ (and this should not be confused with the concept of isometrical embedding of Riemannian manifolds).

²cf. remark 1.1

2.2.1 Radon-Nikodym property

The following result from [1] will come handy:

Theorem 2.7. *Suppose that E is the dual of a separable Banach space. Let $\gamma : [a, b] \rightarrow E$ be a Lipschitz curve; then, by thm. 8.1 in [1], for almost all t there exists the derivative $\dot{\gamma}(t) \in E$ that is defined as*

$$\dot{\gamma}(t) \stackrel{\text{def}}{=} \lim_{\tau \rightarrow 0} \frac{\gamma(t + \tau) - \gamma(t)}{\tau} \quad (11)$$

where the limit is done according to the weak-* topology; and moreover,

$$\|\dot{\gamma}(t)\| = \lim_{\tau \rightarrow 0} \left\| \frac{\gamma(t + \tau) - \gamma(t)}{\tau} \right\| \quad (12)$$

so $\|\dot{\gamma}(t)\|$ coincides with the metric derivative, that is studied in [2].

There follows easily (by applying scalar products to (11)) that

$$\gamma(b) - \gamma(a) = \int_a^b \dot{\gamma}(t) dt \quad (13)$$

and

$$\text{len}^d \gamma = \int_a^b \|\dot{\gamma}(t)\| dt \quad (14)$$

(this last by thm. 4.1.1 in [2]).

It is common to say that E enjoys the *Radon-Nikodym Property*, when the limit in (11) exists in the strong sense, and for almost all t . Note that the *Radon-Nikodym Property* does not hold in general: consider the map $t \mapsto \mathbb{1}_{[t, t+1]}$ in $L^1(\mathbb{R})$, whose derivative should be $t \mapsto \delta_{t+1} - \delta_t$.

We now recall this basilar definition:

Definition 2.8. a Banach space E is **uniformly convex** if $\forall \varepsilon > 0 \exists \delta > 0$,

$$\forall x, y \in E, \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon \implies \|(x + y)/2\| < (1 - \delta).$$

Examples of uniformly convex Banach spaces include $L^p(\Omega, \mathcal{A}, \mu)$ for $p \in (1, \infty)$. Uniformly convex Banach spaces have many interesting properties: for example, they are reflexive (Milman Theorem, III.29 in [4]); moreover, if $x_n \rightarrow x$ in weak sense and $\limsup \|x_n\| \leq \|x\|$ then $x_n \rightarrow x$ in the strong sense (prp III.30 in [4]).

So we obtain a sufficient condition:

Corollary 2.9. *if E is uniformly convex and separable, then it enjoys the Radon-Nikodym Property (indeed eqn. (11) and eqn. (12) imply that the limit in (11) is valid also in the strong sense).*

2.2.2 Embeddings in uniformly convex Banach spaces

If E is uniformly convex then in particular the closed ball $\{x \mid \|x\| \leq 1\}$ is strictly convex; this has a curious implication.

Lemma 2.10. *Suppose the closed balls in E are strictly convex. Consider E as a metric space, with distance $d_E(x, y) = \|x - y\|$. The segment connecting $x, y \in E$ is the unique minimal geodesic (up to reparametrization).*

Proof. We will prove that, for x, y , for any minimal geodesic $\gamma : [0, 1] \rightarrow M$ connecting x to y , if γ is reparametrized to arc parameter then $\gamma(1/2) = (x + y)/2$; iterating this reasoning with finer subdivision we obtain that $\gamma(t) = (tx + (1 - t)y)$.

With no loss of generality, up to translation and scaling, suppose $y = -x$ and $\|x\| = 1$. The segment $t \mapsto tx$ is a geodesic for $t \in [-1, 1]$, by the theorem 2.7, and its length is 2. Suppose now that $\gamma : [-1, 1] \rightarrow M$ is another geodesic: then $\text{len } \gamma = 2$, and, up to reparametrization, $\|\dot{\gamma}\| = 1$ at almost all points; in particular, setting $z = \gamma(0)$, $\|z - y\| \leq 1$ and $\|x - z\| \leq 1$; but then, by triangular inequality, $\|z + x\| = \|x - z\| = 1$. Suppose that $z \neq 0$; then $\|(z + x) - (x - z)\| > 0$; by strict convexity, though, this implies that $\|(z + x) + (x - z)\|/2 = \|x\| < 1$, and this is a contradiction. \square

Theorem 2.11. *Suppose that (M, d) is a complete space, and that $i : M \rightarrow E$ is an isometrical immersion in a uniformly convex Banach space E . If, given $x, y \in M$, $d(x, y) = d^g(x, y)$, then the segment connecting $i(x), i(y)$ is all contained in $i(M)$.*

In particular, if (M, d) is path-metric then $i(M)$ is convex, and then any two points in M can be joined by a unique minimal geodesic (unique up to reparametrization).

Proof. Note that $i(M)$ is complete, and then it is closed in E . We will prove that, for any $x, y \in i(M)$, $(x + y)/2 \in i(M)$; we can then iterate this idea to further subdivide, and since $i(M)$ is closed then this proves the whole segment connecting x, y is in $i(M)$; for the above lemma, the segment is the unique minimal geodesic.

We now fix $x, y \in i(M)$: there must be paths $\gamma_n : [-1, 1] \rightarrow i(M)$ connecting x to y with length $\text{len}(\gamma) < L_n \stackrel{\text{def}}{=} \|x - y\| + 2/n$.

As in the lemma before, we suppose for simplicity that $y = -x$ and $\|x\| = 1$ (so $L_n = 2 + 2/n$); and we reparametrize so that $\|\dot{\gamma}_n\| = 1 + 1/n$: hence setting $z_n = \gamma_n(0)$

$$\|z_n + x\| \leq 1 + 1/n, \quad \|x - z_n\| \leq 1 + 1/n.$$

and then by triangle inequality $\|z_n + x\| \rightarrow 1, \|z_n - x\| \rightarrow 1$. Setting

$$w_n = (z_n + x)/\|z_n + x\|, \quad v_n = (x - z_n)/\|z_n - x\|$$

we can prove that $\|(w_n + v_n)/2\| \rightarrow 1$ hence by the uniform convexity of E we obtain that $w_n - v_n \rightarrow 0$ and then $z_n \rightarrow 0$. Since $z_n \in i(M)$ and $i(M)$ is closed then $0 \in i(M)$. \square

The above is a “rigidity theorem”, in that it restricts the class of metric spaces that can be isometrically embedded in a uniformly convex Banach space E .

Corollary 2.12. *a complete compact finite dimensional Riemannian manifold M cannot be isometrically embedded in a uniformly convex Banach space E : indeed in this space M there are two points that can be joined by more than one minimal geodesic.*

When E is not uniformly convex, on the other hand, strange behaviours arise.

Remark 2.13. Let $L^\infty = L^\infty(\Omega, \mathcal{A}, \mu)$ and suppose Ω is not an atom of μ , that is, suppose the dimension of L^∞ is greater than 1. Given generically $f, g \in L^\infty$, there is an uncountable number of minimal geodesics connecting them.

Proof. We can assume without loss of generality that $g = 0$ and that $\|f\| = 1$. Let $A = \{|f| = 1\}$. We will prove that if there is only one geodesic then $|f| = \mathbb{1}_A$. Indeed if $|f| \neq \mathbb{1}_A$ then $\mu\{|f| < 1\} > 0$. Let $0 < t < 1$ be such that $\mu\{|f| < t\} > 0$;

obviously $\mu\{|f| \geq t\} > 0$ since $\|f\| = 1$; let $A' = \{|f| \geq t\}$ and $A'' = \{|f| < t\}$. Given any diffeomorphism $b : [0, 1] \rightarrow [0, 1]$ with $b'(s) \leq 1/t$,

$$\gamma(t) \stackrel{\text{def}}{=} t f \mathbb{1}_{A'} + b(t) f \mathbb{1}_{A''}$$

is a geodesic. Indeed its derivative is

$$\gamma'(t) \stackrel{\text{def}}{=} f \mathbb{1}_{A'} + b'(t) f \mathbb{1}_{A''}$$

and $\|\gamma'(t)\| = 1$ by construction.

The family of f s.t. $|f| = \lambda \mathbb{1}_A$ is closed and has empty interior. \square

The idea of *isometrical embedding* is quite powerful: indeed any separable metric space may be isometrically embedded in ℓ^∞ (that is the dual of the separable space ℓ^1): so the breadth of application of the theorem 2.7 is general, and is at the basis of many results in [1]. But the embedding in ℓ^∞ that is studied in [1] is not suited for our practical applications:

- it would not respect the geometric properties of the space (as we discussed in sec. 1.2)
- it would be too difficult to find a satisfactory notion of “*shooting of minimal geodesics*” using this embedding.

For all above reasons, we will consider *isometrical embeddings* in this paper as well but we will (for the most interesting applications) use an explicitly chosen embedding in uniformly convex Banach spaces.

2.3 Definitions

We introduce some definitions that will be used in the rest of the paper

We will write $s \vee t = \max\{s, t\}$ and $s^+ = \max\{s, 0\}$, when $s, t \in \mathbb{R}$.

We will write $B(x, r)$ or $B_r(x)$ for the open ball of center x and radius $r > 0$ in \mathbb{R}^N ; we will shortly write B_r for $B_r(0)$. Similarly $D_r(x)$ will be the closed ball of center x and radius $r > 0$ in \mathbb{R}^N , and $D_r = D_r(0)$.

We define the **fattened set** to be

$$A + D_r = \{x + y \mid x \in A, |y| \leq r\} = \bigcup_{x \in A} D_r(x) = \{y \mid u_A(y) \leq r\}.$$

This fattened set is always closed, (since the distance function $u_A(x)$, that was defined in (1), is continuous).

We will say that a family $A_{i \in I}$ of sets in \mathbb{R}^N is *equibounded* if there is a $R > 0$ such that $A_i \subset D_R$ for all i .

We denote by \mathcal{L}^N the N dimensional Lebesgue measure, and $\omega_N \stackrel{\text{def}}{=} \mathcal{L}^N(B_1)$; we write shortly $\int_A f(x) dx$ for the Lebesgue integral.

2.4 Hausdorff distance

A fundamental example of metric on \mathcal{I} is the **Hausdorff distance**

$$d_H(\Omega, \Omega') \stackrel{\text{def}}{=} \inf\{\delta > 0 \mid \Omega' \subset (\Omega + D_\delta), \Omega \subset (\Omega' + D_\delta)\} .$$

It is not difficult to verify that

$$d_H(\Omega, \Omega') = \sup_{x \in \mathbb{R}^N} |u_\Omega(x) - u_{\Omega'}(x)|, \quad (15)$$

see for example Thm. 2.2 in ch. 4 in Delfour–Zolesio [6].

This metric enjoys many important properties.

Theorem 2.14. *The metric space (\mathcal{I}, d_H) satisfies:*

- given $r > 0$, the family of r -bounded compact sets

$$\{\Omega \in \mathcal{I} \mid \Omega \subset D_r\}$$

is compact; in particular, the set

$$\mathbb{D} \stackrel{\text{def}}{=} \{\Omega \mid d_H(\Omega, \Omega') \leq \rho\}$$

is compact;

- (\mathcal{I}, d_H) is path-metric (that is, $d_H = (d_H)^g$)
- consequently, by Prp. 2.2 any two $\Omega, \Omega' \in \mathcal{I}$ may be joined by a minimal geodesics;
- and moreover, by Prp. 2.3, (\mathcal{I}, d_H) is complete.

The first statement is a well known property of the Hausdorff distance, see e.g. [6] pag. 194. By exploiting the characterization (15), it also follows from a diagonal/compactness argument and the following rigidity property:

Lemma 2.15. *Let Ω_n be closed sets, and suppose that $\lim_n u_{\Omega_n}(x) = f(x)$ for all x in a dense subset D of \mathbb{R}^N . Then there is a closed set Ω such that $u_\Omega(x) = f(x)$ for all $x \in D$, and $u_{\Omega_n} \rightarrow u_\Omega$ uniformly on compact sets; moreover if (and only if) Ω_n is equibounded then $u_{\Omega_n} \rightarrow u_\Omega$ uniformly.*

Proof. The proof may follow from the theory of Viscosity Solutions: it is well known, indeed, that u_Ω is the unique solution to a properly defined *Eikonal equation*; and that viscosity solutions do enjoy the required rigidity property.

We propose here instead a direct proof. We set $u_n \stackrel{\text{def}}{=} u_{\Omega_n}$; it is easily proved that u_n is 1-Lipschitz, that is

$$|u_n(x) - u_n(y)| \leq |x - y| \quad \forall x, y \quad (16)$$

so passing to the limit in the above (16), we obtain

$$|f(x) - f(y)| \leq |x - y| \quad \forall x, y \in D \quad (17)$$

and then there is a unique extension of f to a positive function $g : \mathbb{R}^N \rightarrow \mathbb{R}$ that is again 1-Lipschitz, that is,

$$|g(x) - g(y)| \leq |x - y| \quad \forall x, y. \quad (18)$$

It is easy to prove that $u_n(x) \rightarrow g(x)$ for all x , and actually (by imitating the proof of Ascoli–Arzelà theorem) that $u_n \rightarrow g$ uniformly on compact sets.

Let $\Omega = \{g = 0\}$; to conclude the proof, we need to prove that $g = u_\Omega$. To this end, we first prove that $g \geq u_\Omega$: indeed, fixing x , $u_n(x) = |x - y_n|$ for at least one point $y_n \in \Omega_n$; since $u_n(x) \rightarrow g(x)$, then the sequence $\{y_n\}$ is bounded, so (up to a subsequence n_k) it converges to a point y ; since the family u_n is 1-Lipschitz and $u_n(y_n) = 0$ then $g(y) = 0$, that is $y \in \Omega$: hence

$$g(x) = \lim_k u_{n_k}(x) = \lim_k |y_{n_k} - x| = |y - x| \geq u_\Omega(x).$$

Conversely, let $y \in \Omega$ be such that $u_\Omega(x) = |x - y|$; then by (18) $g(x) \leq g(y) + |x - y| = |x - y| = u_\Omega(x)$.

To conclude, supposing that Ω_n is equibounded, then choosing $R > 0$ such that $\Omega_n \subset D_R$, we know that $u_{\Omega_n} \rightarrow u_\Omega$ uniformly on D_R , so given $\varepsilon > 0$ for n large $|u_n - u| < \varepsilon$ in D_R and then

$$u_n(x) = \inf_{y \in D_R} (|x - y| + u_n(y)) \leq \inf_{y \in D_R} (|x - y| + u(y) + \varepsilon) = u(x) + \varepsilon$$

where the first and last equalities are due to the Dynamical Programming principle; and similarly we obtain that $u(x) \leq u_n(x) + \varepsilon$. The “only if” part follows from (15). \square

To prove the above second property in 2.14, we may use the first property and the following *Menger convexity* result

Proposition 2.16. *Let $A, B \in \mathcal{I}$ be two compact sets, then for all $\lambda \in [0, 1]$ there exists a compact set C such that*

- $d_H(A, C) = \lambda d_H(A, B)$,
- $d_H(B, C) = (1 - \lambda) d_H(A, B)$.

Proof. We write $[A]_r = A + D_r$ for the fattened set. Let $\mu = d_H(A, B)$. We consider the set

$$C \stackrel{\text{def}}{=} \{z | \exists x \in A, y \in B, |x - y| \leq \mu, |x - z| \leq \lambda\mu, |y - z| \leq (1 - \lambda)\mu\}.$$

We prove that the set C has the properties we need. In particular it is enough to prove only the first one because of the symmetry in the two conditions. If $x \in A$ then there exists $y \in B$ such that $|x - y| \leq \mu$, then the a point $z = (1 - \lambda)x + \lambda y$ satisfies

$$|x - z| \leq \lambda\mu, |y - z| \leq (1 - \lambda)\mu.$$

Such a z must be an element of C and so we found an element of C with distance less or equal to $\lambda\mu$. This means that $x \in [C]_{\lambda\mu}$ and it is true for all $x \in A$ so $A \subset [C]_{\lambda\mu}$.

Let's take now $z \in C$. From the definition of the elements of C we have that there must exists $x \in A$ and $y \in B$ such that $|x - z| \leq \lambda\mu$, $|y - z| \leq (1 - \lambda)\mu$. This means that $z \in [A]_{\lambda\mu}$. This is true for all $z \in C$ so $C \subset [A]_{\lambda\mu}$.

To finish the proof we have to show that the set C is compact. It is clearly bounded because it is contained by $[A]_{\lambda\mu}$. We have to show that it is closed. Suppose we have a sequence $\{z_k\}_k \subset C$ such that $z_k \rightarrow z$. Then for each z_k we can find two elements $x_k \in A, y_k \in B$ with the properties:

$$|x_k - z_k| \leq \lambda\mu, |y_k - z_k| \leq (1 - \lambda)\mu.$$

The sets A and B are compacts so we can chose a subsequence (for simplicity we use the same index k) such that $x_k \rightarrow x \in A$ and $y_k \rightarrow y \in B$. It is obvious to see that the points x, y, z satisfy the following inequalities:

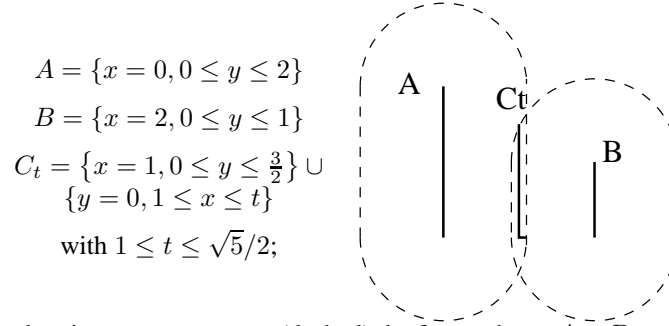
$$|x - y| \leq \mu, |x - z| \leq \lambda\mu, |y - z| \leq (1 - \lambda)\mu.$$

This means that z is an element of C and this concludes the proof. \square

Unfortunately (\mathcal{I}, d_H) is quite “unsmooth”, as shown by this example (that is similar to 2.13 – and for a reason!).

Example 2.17. There are choices of $\Omega, \Omega' \in \mathcal{I}$ that may be joined by an uncountable number of geodesics.

In fact we can consider this simple example:



and in the picture we represent (dashed) the fattened sets $A + B_{\sqrt{5}/2}$ and $B + B_{\sqrt{5}/2}$. Note that $d_H(A, B) = \sqrt{5}$ while $d_H(A, C_t) = d_H(B, C_t) = \sqrt{5}/2$: so C_t are all midpoints that are on different geodesics between A and B .

We conclude with a family of nice properties.

- Proposition 2.18.**
1. The fattening map $\lambda \mapsto A + D_\lambda$ is Lipschitz (of constant one).
 2. Given $\lambda > 0$, the “fattened area map” $L_\lambda(A) \stackrel{\text{def}}{=} \mathcal{L}^N(A + D_\lambda)$ is continuous.
 3. Consequently, the area map $L(A) \stackrel{\text{def}}{=} \mathcal{L}^N(A)$ is upper semi continuous.
 4. Let $\# : \mathcal{I} \rightarrow \mathbb{N} \cup \infty$ be the number $\#\Omega$ of connected components of a closed set Ω . Then $\#$ is lower semi continuous in the metric space (\mathcal{I}, d_H) .

As a corollary, the family of connected compact sets is a closed family in (\mathcal{I}, d_H) .

Proof. 1. Obvious.

2. if $A_n \rightarrow A$ then for fixed $\varepsilon > 0$ and definitively in n ,

$$A_n \subset A + D_\varepsilon, \quad A \subset A_n + D_\varepsilon$$

and then

$$A_n + D_\lambda \subset A + D_{\varepsilon+\lambda}, \quad A + D_{\lambda-\varepsilon} \subset A_n + D_\lambda$$

passing to Lebesgue measures,

$$\mathcal{L}^N(A + D_{\lambda-\varepsilon}) \leq \liminf_n \mathcal{L}^N(A_n + D_\lambda) \leq \limsup_n \mathcal{L}^N(A_n + D_\lambda) \leq \mathcal{L}^N(A + D_{\varepsilon+\lambda})$$

and we let $\varepsilon \rightarrow 0$.

3. Since it is the pointwise limit $L_\lambda(A) \downarrow L(A)$ for $\lambda \rightarrow 0$.

4. See Thm. 2.3 in ch. 4 in [6].

□

3 L^p -like metrics of shapes

The definition of the Hausdorff distance by eqn. (15) leads us back to the paradigm of *representation/embedding*; but in this case it is unfortunately not precise, since the Banach metric that we use, namely

$$\|f\| = \|f\|_\infty \stackrel{\text{def}}{=} \sup_x |f(x)|$$

is usually associated to the spaces $C_b(\mathbb{R}^N)$ of bounded functions — whereas the distance function u_A is not bounded! What follows is a simple yet effective workaround.

Definition 3.1. We fix $p \in [1, \infty]$; we fix a function $\varphi : [0, \infty) \rightarrow (0, \infty)$ monotonically decreasing and of class C^1 , such that

$$\varphi(|x|) \in L^p(\mathbb{R}^N). \quad (19)$$

Note that, for $p < \infty$, the above is equivalent to asking that

$$\int_0^\infty t^{N-1} \varphi(t)^p dt < \infty \quad (20)$$

and it implies that $\lim_{t \rightarrow \infty} \varphi(t) = 0$; for $p = \infty$ we instead ask that $\lim_{t \rightarrow \infty} \varphi(t) = 0$ as an extra hypothesis.

An example of such a function is $\varphi(t) = \exp(-t)$, or $\varphi = (1+t)^{-(N+1)/p}$.

We will often write $v_A = \varphi \circ u_A$ for simplicity.

Lemma 3.2. Let $\Omega \subset \mathbb{R}^N$ be closed and non empty; suppose $p < \infty$; then

(a) $v_\Omega \in L^p(\mathbb{R}^N)$ if and only if

(b) Ω is bounded (and then Ω is compact).

Proof. We first prove that (a) \implies (b) by contradiction. Let us assume that Ω is unbounded. Then there exists a sequence $\{x_k\} \subset \Omega$ such that $|x_k| \rightarrow \infty$ and $d(x_k, x_q) > 2$ for all $k, q \in \mathbb{N}, k \neq q$. The sequence of sets $B_1(x_k)$ is disjoint. It is easy to see that $v_\Omega(x) > \varphi(1)$ for $x \in \bigcup_k B_1(x_k)$, and then $v \notin L^p$.

Then we prove that (b) \implies (a). If Ω is bounded we can find a ball B_R such that $\Omega \subset B_R$. Then easily we have $u_\Omega \geq u_B \implies v_\Omega \leq v_B$, but $v_B \in L^p$ (as is easily proved by $v_B(x) = \varphi(|x| - R)^+$ and by (20)) and then also $v_\Omega \in L^p$. □

Definition 3.3. Given $A, B \in \mathcal{I}$, we define

$$d_{p,\varphi}(A, B) \stackrel{\text{def}}{=} \|\varphi(u_A) - \varphi(u_B)\|_{L^p(\mathbb{R}^N)}$$

By the above lemma, this distance is finite. We will often write d for $d_{p,\varphi}$ in the following, for simplicity.

The above distance is obtained by the *representation* of a shape A as v_A , combined with the *embedding* of v_A in $L^p(\mathbb{R}^N)$. For this reason, we may identify our shape space with

$$N_c \stackrel{\text{def}}{=} \{v_\Omega \mid \Omega \in \mathcal{I}\} \quad (21)$$

that is a subspace of L^p .

Remark 3.4. By the definition of d , the map $\Omega \mapsto v_\Omega$ is an isometrical embedding of \mathcal{I} inside L^p , and the image is N_c ; N_c is a closed subset of L^p , by the completeness result 3.11 that we will prove in the following.

We will exploit this embedding in the following, as in §3.6.

It is immediate to verify that $d_{p,\varphi}$ satisfies these properties.

- The embedding $A \mapsto v_A$ is injective: if $v_A \stackrel{\sim}{=} v_B$ then $u_A \stackrel{\sim}{=} u_B$ (since φ is monotonically decreasing, and so it is injective); but, by lemma 2.15, this implies that $u_A = u_B$ and then $A = B$; consequently, for all $A, B \in \mathcal{I}$, $d_{p,\varphi}(A, B) = 0$ iff $A = B$.
- $d_{p,\varphi}$ is euclidean invariant, as we requested in sec. 1.2.
-

$$d_{p,\varphi}(\Omega_1, \Omega_2) < \|v_{\Omega_1}\|_{L^p} + \|v_{\Omega_2}\|_{L^p}. \quad (22)$$

for $p < \infty$, and

$$d_{\infty,\varphi}(\Omega_1, \Omega_2) < \varphi(0).$$

Proof. When $p < \infty$, by the Minkowski inequality we have that $d(\Omega_1, \Omega_2) \leq \|v_{\Omega_1}\|_{L^p} + \|v_{\Omega_2}\|_{L^p}$; moreover equality would hold only if $v_{\Omega_1} = -v_{\Omega_2}$ and this is impossible; when $p = \infty$ we use the fact that $\varphi > 0$. \square

- (*Separation at infinity*) given two bounded sets Ω_1, Ω_2 we have

$$\lim_{|\tau| \rightarrow \infty} d_{p,\varphi}(\Omega_1, \Omega_2 + \tau) = \|v_{\Omega_1}\|_{L^p} + \|v_{\Omega_2}\|_{L^p}; \quad (23)$$

for $p < \infty$, and

$$\lim_{|\tau| \rightarrow \infty} d_{\infty,\varphi}(\Omega_1, \Omega_2 + \tau) = \varphi(0); \quad (24)$$

Proof. For the case $p < \infty$ this comes from a general result for L^p functions; for $p = \infty$ it derives from the hypothesis $\lim_{t \rightarrow \infty} \varphi(t) = 0$. \square

- (*Scaling*) If $p < \infty$ and $\lambda > 0$ is a rescaling of the space, then the rescaled distance may be expressed as

$$d_{p,\varphi}(\lambda\Omega_1, \lambda\Omega_2) = \lambda^{N/p} d_{p,\tilde{\varphi}}(\Omega_1, \Omega_2) \quad (25)$$

where $\tilde{\varphi}(r) = \varphi(\lambda r)$; indeed

$$d_{p,\varphi}(\lambda\Omega_1, \lambda\Omega_2)^p = \int |v_{\lambda\Omega_1}(x) - v_{\lambda\Omega_2}(x)|^p dx \quad (26)$$

$$= \lambda^N \int |v_{\lambda\Omega_1}(\lambda z) - v_{\lambda\Omega_2}(\lambda z)|^p dz \quad (27)$$

$$= \lambda^N \int |\varphi(\lambda u_{\Omega_1}(z)) - \varphi(\lambda u_{\Omega_2}(z))|^p dz \quad (28)$$

$$= \lambda^N d_{p,\tilde{\varphi}}(\Omega_1, \Omega_2)^p \quad (29)$$

where to go from (26) to (27) we used the change of variable $x = \lambda z$ and the property of the distance function

$$u_{\lambda\Omega}(\lambda z) = \lambda u_{\Omega}(z) \quad (30)$$

to change (27) to (28).

Remark 3.5. The inequality (22) easily implies that the balls of the distance d in general are not compact sets. Indeed it is enough to consider a set Ω and the following ball: $\mathbb{D} = \{A \mid d(A, \Omega) \leq 2r\}$ with $r = \|v_{\Omega}\|_{L^p}$. Then the sequence: $\{\Omega + n\tau\}_{n \in \mathbb{N}}$ with $\tau \in \mathbb{R}^N \setminus \{0\}$ is contained in \mathbb{D} and it does not have any convergent subsequence.

To continue with our study of d , we prove this fundamental inequality.

Lemma 3.6 (Local equiboundedness). *There is a continuous and increasing function $b : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $b(0) = 0$ such that, for any $\Omega, \Omega' \in \mathcal{I}$ satisfying*

$$\|v_{\Omega} - v_{\Omega'}\|_{L^p} < b(r),$$

then $\Omega' \subset \Omega + D_r$.

Proof. Set $K \stackrel{\text{def}}{=} \Omega + D_r$. It is easy to check that

$$v_K(x) = \varphi((u_{\Omega}(x) - r)^+).$$

To prove the proposition for $p \in [1, \infty)$, suppose that $x_0 \in \Omega'$, but $x_0 \notin K$; for $y \in B(x_0, r/2)$ recall the simple triangular inequality

$$u_{\Omega}(y) \geq r - |x_0 - y| \geq |x_0 - y| \geq u_{\Omega'}(y)$$

hence

$$v_{\Omega}(y) \leq \varphi(r - |x_0 - y|) \leq \varphi(|x_0 - y|) \leq v_{\Omega'}(y)$$

$$\begin{aligned} \|v_{\Omega} - v_{\Omega'}\|_{L^p}^p &\geq \int_{B(x_0, r/2)} |v_{\Omega'} - v_{\Omega}|^p dx \geq \\ &\geq \int_{B(x_0, r/2)} |\varphi(|x_0 - y|) - \varphi(r - |x_0 - y|)|^p dx = b(r)^p \end{aligned}$$

where

$$b(r)^p \stackrel{\text{def}}{=} \omega_N N \int_0^{r/2} t^{N-1} (\varphi(t) - \varphi(r-t))^p dt$$

where ω_N is the volume of the ball B_1 . It is easy to prove that b is continuous and increasing (by direct derivation); that $b(0) = 0$ and that $\lim_{r \rightarrow \infty} b(r) = \|\varphi(|x|)\|_{L^p}$.

The case $p = \infty$ is simpler: in this case we can note that

$$\|v_\Omega - v_{\Omega'}\|_\infty \geq v_{\Omega'}(x_0) - v_\Omega(x_0) \geq \varphi(0) - \varphi(r)$$

and set $b(r) = \varphi(0) - \varphi(r)$. \square

Corollary 3.7. *As a corollary we obtain that for $d(\Omega, \Omega')$ small enough,*

$$d_H(\Omega, \Omega') \leq b^{-1}\left(d(\Omega, \Omega')\right).$$

Remark 3.8. The above does not hold for arbitrarily large distance $d(\Omega, \Omega')$: indeed, let $\Omega = \{0\}$ and $\Omega_n = \{ne_1\}$: then $d(\Omega, \Omega_n) \rightarrow 2\|\varphi(|x|)\|_{L^p}$ (as we mentioned in (23)).

We can also obtain a converse inequality, as follows

Lemma 3.9. *There is a family of continuous functions $f_R : [0, 1] \rightarrow \mathbb{R}^+$ with $f_R(0) = 0$, such that, for any $\Omega, \Omega' \in \mathcal{I}$, if Ω has diameter R and $d_H(\Omega, \Omega') < 1$, then*

$$d(\Omega, \Omega') \leq f_R\left(d_H(\Omega, \Omega')\right).$$

Proof. We provide the proof for $p < \infty$. Note that if Ω has diameter R and $d_H(\Omega, \Omega') < 1$, then Ω' has diameter at most $R + 2$. Up to translation, suppose that B_{2R+4} contains both Ω and Ω' : then

$$v_\Omega(x), v_{\Omega'}(x) \leq \varphi(|x| - 2R - 4)^+$$

so

$$\int_{\mathbb{R}^N \setminus B_r} |v_\Omega(x) \vee v_{\Omega'}(x)|^p dx \leq a_R(r)$$

where

$$a_R(r) \stackrel{\text{def}}{=} \int_{\mathbb{R}^N \setminus B_r} \varphi(|x| - 2R - 4)^p dx = \omega_N N \int_r^\infty t^{N-1} \varphi(t - 2R - 4)^p dt$$

and note that $a_R(r) \rightarrow 0$ for $r \rightarrow \infty$. At the same time, let $l(r) = \sup_{[0, r]} |\varphi'|$: then

$$\forall x \in B_r, \quad |v_\Omega(x) - v_{\Omega'}(x)| \leq l(r + 4 + 2R) |u_\Omega(x) - u_{\Omega'}(x)|$$

so

$$\begin{aligned} \int_{B_r} |v_\Omega(x) - v_{\Omega'}(x)|^p dx &\leq \omega_N r^N l(r + 4 + 2R)^p \sup_{x \in B_r} |u_\Omega(x) - u_{\Omega'}(x)|^p \leq \\ &\leq \omega_N r^N l(r + 4 + 2R)^p d_H(\Omega, \Omega')^p \end{aligned}$$

Summarizing,

$$\begin{aligned} d(\Omega, \Omega')^p &= \int_{\mathbb{R}^N \setminus B_r} |v_\Omega(x) - v_{\Omega'}(x)|^p dx + \int_{B_r} |v_\Omega(x) - v_{\Omega'}(x)|^p dx \leq \\ &\leq a_R(r) + \omega_N r^N l(r + 4 + 2R)^p d_H(\Omega, \Omega')^p \end{aligned}$$

Let eventually

$$g_R(s) = \inf_{r \geq 2R+4} [a_R(r) + \omega_N r^N l(r + 4 + 2R)^p s]$$

and note that it is concave and that $\lim_{s \rightarrow 0} g_R(s) = 0$; and let $f_R(s) = \sqrt[p]{g_R(s^p)}$. \square

Combining the two lemmas 3.9 and 3.7, we obtain that

Theorem 3.10. *The topology induced by d over the space \mathcal{I} is equivalent to the one induced by d_H .*

This implies that all properties of the Hausdorff distance listed in proposition 2.18 are valid for the distance d as well.

3.1 Completeness and compactness

By prop. 3.10, we know that (\mathcal{I}, d) is locally compact.

We now prove that it is complete:

Proposition 3.11 (Completeness). *The space (\mathcal{I}, d) is complete.*

Proof. Let Ω_n be a Cauchy sequence; this means that, $\{v_{\Omega_n}\}_n \subset N_c$ is a Cauchy sequence: since L^p is complete, $v_{\Omega_n} \rightarrow g$ in L^p . It is well known (see e.g. thm IV.9 in [4]) that, up to subsequence that we indicate with $\{v_k\}_k$, there is also convergence $v_k(x) \rightarrow g(x)$ for almost all x ; let $u_k(x) \stackrel{\text{def}}{=} \varphi^{-1}v_k(x)$ and $u = \varphi^{-1}g$; then $u_k(x) \rightarrow u(x)$ on a dense subset, so by the lemma 2.15, $u = u_\Omega$ where $\Omega \stackrel{\text{def}}{=} \{u = 0\}$. \square

Summarizing, this and 3.10 imply that N_c is a complete (that is, closed) and locally compact subset of L^p .

Remark 3.12. The above implies an interesting property of the subset N_c of L^p : it admits a small neighbourhood U on L^p such that, for $f \in U$, there is at least a $v \in N_c$ providing the minimum of the distance $\inf_{v \in N_c} \|f - v\|$. As far as we know, this minimum may fail to be unique.

3.2 Shape analysis

The family of distances is suitable for Shape Analysis: we can indeed prove

Proposition 3.13. *Let $G = \mathcal{O}(N) \ltimes \mathbb{R}^N$ be the Euclidean group of rotation and translation; as in (10), we can define the quotient metric by*

$$d_q([A], [B]) = \inf_{g \in G} d(gA, B). \quad (31)$$

Then the above infimum is a minimum; so $d_q([A], [B]) > 0$ when $[A] \neq [B]$.

Proof. Choose a minimizing sequence $\{g_n = (R_n, T_n)\}_{n \in \mathbb{N}}$, that is

$$\inf_{g \in G} d(gA, B) = \lim_{n \rightarrow \infty} d(g_n A, B) = \lim_{n \rightarrow \infty} d(R_n A + T_n, B).$$

Then $\{T_n\}_{n \in \mathbb{N}}$ must be bounded; in fact, let us assume by contradiction that $|T_n| \rightarrow \infty$, then by (22) we would have that

$$d(A, B) < \|v_A\|_{L^p} + \|v_B\|_{L^p}$$

and by (23) that

$$\lim_{n \rightarrow \infty} d(R_n A + T_n, B) = \|v_A\|_{L^p} + \|v_B\|_{L^p},$$

so $\{g_n\}$ is not a minimizing sequence. This contradiction is generated by the assumption that $\{T_n\}$ is unbounded; then the translation part of every minimizing sequence of (31) must be bounded. By compactness we have that there exists a limit transformation $g = (R, T) \in K$ such that $g_n \rightarrow g$ and by continuity of $d(fA, B)$ with respect of $f \in G$, we have that $d(gA, B) = d^g([A], [B])$. \square

3.3 d^g and geodesics

In this section we restrict $p \in (1, \infty)$.

Unfortunately $d = d_{p,\varphi}$ is not path-metric:

Proposition 3.14. *Given any two $A, B \in \mathcal{I}$ with $A \neq B$*

- *then there is at most one $\lambda \in (0, 1)$ such that $\lambda v_A + (1 - \lambda)v_B \in N_c$*
- *consequently, by thm. 2.11, we have that $d(A, B) < d^g(A, B)$.*

Proof. It is immediate to show that $f_\lambda = \lambda v_A + (1 - \lambda)v_B$ assumes the value $\varphi(0)$ only on the intersection of the two sets $A \cap B$ for any $\lambda \in (0, 1)$. Then $f_\lambda \in N_c$ implies that $f_\lambda = v_{A \cap B}$. Let $x \in (A \cap B)^c$ such that $v_A(x) \neq v_B(x)$. We have that $f_{\lambda_1}(x) \neq f_{\lambda_2}(x)$ for any $\lambda_1, \lambda_2 \in (0, 1)$ and $\lambda_1 \neq \lambda_2$. Then there is at most one $\lambda \in (0, 1)$ such that $f_\lambda(x) = v_{A \cap B}(x)$. \square

So, to prove that the metric d admits minimal geodesics, we have to study d^g as well; to this end, we prove two results.

Proposition 3.15. *If*

$$\varphi'(|x|) \in L^p(\mathbb{R}^N) \quad (32)$$

then the space $(\mathcal{I}, d_{p,\varphi})$ is Lipschitz-arc connected.

Proof. Indeed, let $\gamma(t) = t\Omega$ be the path that rescales Ω to the singleton $\{0\}$; we prove that γ is Lipschitz.

It is not difficult to prove that the map $(t, x) \mapsto u_{t\Omega}(x)$ is jointly Lipschitz. Then $u_{t\Omega}(x)$ is differentiable at almost all t, x , and fix such a t, x ; note that

$$u_{t\Omega}(x) = tu_\Omega\left(\frac{x}{t}\right)$$

(as in eqn. (30)); hence, taking derivatives w.r.t. x we obtain

$$\nabla u_{t\Omega}(x) = \nabla u_\Omega\left(\frac{x}{t}\right)$$

while taking derivatives w.r.t. t we obtain

$$\partial_t u_{t\Omega}(x) = u_\Omega\left(\frac{x}{t}\right) - \frac{1}{t} \langle \nabla u_\Omega\left(\frac{x}{t}\right) \cdot x \rangle = \frac{1}{t} (u_{t\Omega}(x) - \langle \nabla u_{t\Omega}(x) \cdot x \rangle) .$$

Suppose now that $x \notin t\Omega$ and let $y \in t\Omega$ be a minimum distance point from x : then

$$u_{t\Omega}(x) = |x - y| \quad , \quad \nabla u_{t\Omega}(x) = \frac{x - y}{|x - y|}$$

so

$$\begin{aligned}\partial_t u_{t\Omega}(x) &= \frac{1}{t} \left(|x - y| - \left\langle \frac{x - y}{|x - y|} \cdot x \right\rangle \right) = \\ &= -\frac{1}{t|x - y|} \langle x - y \cdot y \rangle = -\left\langle \frac{x - y}{|x - y|} \cdot \frac{y}{t} \right\rangle\end{aligned}\quad (33)$$

so if $\Omega \subset B_r$ we obtain that $|\partial_t u_{t\Omega}(x)| \leq r$. If instead $x \in t\Omega$ and $u_{t\Omega}(x)$ is differentiable at x then $\nabla u_{t\Omega}(x) = 0$ and $\partial_t u_{t\Omega}(x) = 0$.

To conclude (cf. 2.7) we compute

$$\|\dot{\gamma}\|_{L^p}^p = \int |\varphi'(u_{t\Omega}(x))|^p |\partial_t u_{t\Omega}(x)|^p dx \leq r^p \int |\varphi'(u_{t\Omega}(x))|^p dx \quad (34)$$

and we argument as in 3.2. By Rem. 1.1.3 in [3], we conclude that γ is Lipschitz. \square

Remark 3.16. Asking that φ satisfy both (19) and (32) is equivalent to asking that $\varphi(|x|) \in W^{1,p}$. By using the equality in (34) and in (33), it is possible to show that, for most compact sets, the rescaling is a Lipschitz path if and only if $\varphi(|x|) \in W^{1,p}$.

When \mathcal{I} is Lipschitz-arcwise connected, the induced metric $d^g = (d_{p,\varphi})^g$ is a finite metric.

We can prove an equiboundedness result for d^g (that is stronger than 3.6)

Proposition 3.17. *Fix a compact nonempty set Ω , and $r > 0$; then there is a K compact large such that for any closed set Ω' satisfying $d^g(\Omega, \Omega') < r$, then $\Omega' \subset K$.*

Proof. Let $b(r)$ be defined in 3.6. Let $d^g(\Omega, \Omega') < r$, and $\gamma : [0, 1] \rightarrow N_c$ be a Lipschitz path (of constant L) connecting $\gamma(0) = \Omega$ to $\gamma(1) = \Omega'$ such that

$$\text{len } \gamma \leq d^g(\Omega, \Omega') + 1$$

up to reparametrization, we also assume that $L \leq r + 2$. Let n be large, so that $(r + 2)/n \leq b(r)$, and let $K = \Omega + D_{rn}$ (note that n only depends on r). Let $A_i = \gamma(i/n)$ for $i = 0, \dots, n$; we know that

$$d(A_i, A_{i+1}) \leq d^g(A_i, A_{i+1}) \leq L/n < (r + 2)/n \leq b(r)$$

since γ is L -Lipschitz; so we apply recursively the proposition 3.6 on each A_i : we obtain that,

$$A_{i+1} \subset A_i + D_r$$

hence $\Omega' \subset \Omega + D_{rn} = K$. \square

The above results have many interesting consequences:

Theorem 3.18. *if*

$$\varphi'(|x|) \in L^p$$

then for any $\rho > 0$,

$$\mathbb{D}^g(A, \rho) \stackrel{\text{def}}{=} \{A \mid d^g(A, B) \leq \rho\}$$

is compact in the (\mathcal{I}, d) topology; so

- *we obtain by Prp. 2.2 that minimal geodesics do exist;*
- *and by 2.4 that the Geodesic Distance Based Averaging*

$$\bar{A} = \operatorname{argmin}_A \sum_{j=1}^n d^g(A, A_j)^2 \quad (35)$$

of any given collection A_1, \dots, A_n exists.

3.4 Variational description of geodesics

In this section we restrict $p \in (1, \infty)$. If $\gamma(t)$ is a Lipschitz path in N_c , then it is associated to a function $f(t, x) = v_{\gamma(t)}(x)$.

Proposition 3.19. *Suppose that $t \mapsto f(t, \cdot)$ is a Lipschitz path from $t \in [0, 1]$ to $L^p(\mathbb{R}^N)$; then, by 2.9, for almost all t , f admits strong derivative $\frac{df}{dt}$ in $L^p(\mathbb{R}^N)$ (as was defined in eqn. (11)). Moreover*

- f admits weak partial derivative $\partial_t f$, and $\partial_t f = \frac{df}{dt}$ for almost all t .
- If f admits a pointwise partial derivative h for almost all t, x , then $\partial_t f = h$.

Proof. We extend $f(t, x) = f(1, x)$ for $t > 1$, and $f(t, x) = f(0, x)$ for $t < 0$; note that the extended $f(t, \cdot)$ is still Lipschitz in $L^p(\mathbb{R}^N)$; then we define

$$g_\tau(t, x) \stackrel{\text{def}}{=} \frac{f(t + \tau, x) - f(t, x)}{\tau}$$

so

$$\|g_\tau(t, x)\|_{L^p(\mathbb{R}^N)} \leq c$$

where c is the Lipschitz constant of $f(t, \cdot)$; hence

$$\int_0^1 \int_{\mathbb{R}^N} |g_\tau(t, x)|^p dx dt \leq c^p$$

This means that the family g_τ is bounded in $L^p([0, 1] \times \mathbb{R}^N)$, so we can find a sequence $\tau_n \rightarrow 0$ such that $g_{\tau_n} \rightarrow w$ weakly, i.e.

$$\lim_n \int_0^1 \int_{\mathbb{R}^N} g_{\tau_n}(t, x) \psi(t, x) dx dt = \int_0^1 \int_{\mathbb{R}^N} w(t, x) \psi(t, x) dx dt$$

for all $\psi \in C_c^\infty([0, 1] \times \mathbb{R}^N)$. But

$$\int_0^1 \int_{\mathbb{R}^N} g_\tau(t, x) \psi(t, x) dx dt = \int_0^1 \int_{\mathbb{R}^N} f(t, x) \frac{\psi(t - \tau, x) - \psi(t, x)}{\tau} dx dt$$

hence

$$\lim_n \int_0^1 \int_{\mathbb{R}^N} g_{\tau_n}(t, x) \psi(t, x) dx dt = - \int_0^1 \int_{\mathbb{R}^N} f(t, x) \partial_t \psi(t, x) dx dt$$

by dominated convergence, so we conclude that f admits weak derivative, and the derivative is w . The relationship (13) in $L^p(\mathbb{R}^N)$, that is

$$f(b, \cdot) - f(a, \cdot) = \int_a^b \frac{df}{dt} dt$$

implies that

$$\int_a^b \xi \frac{df}{dt} dt = - \int_a^b \frac{d\xi}{dt} f dt$$

for all $\xi \in C_c^\infty([0, 1])$; but then setting $\psi(t, x) = \xi(t)$, we obtain that $\frac{df}{dt} = \partial_t f$. \square

This means that, for almost all t , we can represent the “abstract” derivative $\frac{d\gamma}{dt}$ by means of the weak derivative $\partial_t f(t, \cdot) \in L^p(\mathbb{R}^N)$.

We use this result and eqn. (14) to express the length:

$$\text{len}^d \gamma = \int_0^1 \|\dot{\gamma}(t)\| dt = \int_0^1 \|\partial_t f(t, x)\|_{L^p} dt \quad (36)$$

So to find the minimal geodesic between two compact sets A, B , we need to minimize the above, with the constraint that $f(0, \cdot) = v_A$, $f(1, \cdot) = v_B$, and, for any fixed t , $\varphi^{-1}f(t, \cdot)$ is a distance function.

It is possible to prove (using a reparametrization lemma and Höelder inequality) that the geodesic is also the minimum of the *action*

$$J(\gamma) = \int_0^1 \|\partial_t f(t, x)\|_{L^p}^p dt = \int_0^1 \int_{\mathbb{R}^N} |\partial_t f(t, x)|^p dx dt$$

Equivalently, setting $g(t, x) = u_{\gamma(t)}(x)$, to find geodesics we can minimize

$$J(\gamma) = \int_a^b \int_{\mathbb{R}^N} |\varphi'(\varphi(g)) \partial_t g(t, x)|^p dx dt$$

with the constraint that $g(0, \cdot) = u_A$, $g(1, \cdot) = u_B$, and, for any fixed t , $g(t, \cdot)$ is a distance function.

3.5 Tangent bundle

Let $p \in (1, \infty)$. We identify \mathcal{I} with $N_c \subset L^p$, as by remark 3.4.

Given a $v \in N_c$, let $T_v N_c \subset L^p$ be the contingent cone

$$T_v N_c \stackrel{\text{def}}{=} \left\{ \lim_n t_n (v_n - v) \mid t_n > 0, v_n \in N_c, v_n \rightarrow v \right\} = \left\{ \lambda \lim_n \frac{v_n - v}{\|v_n - v\|_{L^p}} \mid \lambda \geq 0, v_n \rightarrow v \right\},$$

where it is intended that the above limits are in the sense of strong convergence in L^p .

According to theorem 2.7 if $\gamma : [a, b] \rightarrow N_c$ is a Lipschitz curve then $\dot{\gamma}(t) \in T_{\gamma(t)} N_c$ for almost all t .

In the following example we write explicitly the element of the contingent cone relative to a particular curve.

Example 3.20. We fix $\Omega \in \mathcal{I}$, and define the fattening $\Omega_t = \Omega + D_t$ for $t \geq 0$. We are interested in evaluating the derivative $\dot{\gamma}(t)$. As previously done, we use the fact that

$$u_{\Omega_t}(x) = (u_{\Omega}(x) - t)^+ \quad (37)$$

and note that this map is jointly Lipschitz in (t, x) : hence both $u_{\Omega_t}(x)$ and $v_{\Omega_t}(x)$ are almost everywhere differentiable. The pointwise derivative is given by:

$$w = \lim_{\tau \rightarrow 0} \frac{1}{\tau} [v_{\Omega_{t+\tau}} - v_{\Omega_t}] = \begin{cases} -\varphi'(u_{\Omega}(x) - t) & \text{for } x \notin \Omega_t, \\ 0 & \text{for } x \in \overset{\circ}{\Omega}_t. \end{cases} \quad (38)$$

(note that the derivative may not exist for $x \in \partial\Omega_t$). If $\varphi'(|x|) \in L^p$ then $w \in L^p$, and it can be shown that

$$w = \lim_{\tau \rightarrow 0} \frac{1}{\tau} [v_{\Omega_{t+\tau}} - v_{\Omega_t}]$$

in the L^p sense; then w is in the contingent cone. In particular, by Rem. 1.1.3 in [3], we obtain that the curve γ is Lipschitz for $t \in [0, T]$.

Unfortunately the contingent cone is not capable of expressing some shape motions

Example 3.21. We consider the *removing* motion; to simplify the matter, let A be compact, and suppose that the origin 0 is in the internal part of A ; let $A_t \stackrel{\text{def}}{=} A \setminus B_t$ be the removal of a small ball from A : then we can explicitly compute (for $r > 0, s > 0$ small)

$$\begin{aligned} \|v_{A_{r+s}} - v_{A_s}\|_{L^p}^p &= \omega_N N \int_s^{r+s} t^{N-1} \left(\varphi(0) - \varphi(s+r-t) \right)^p dt + \\ &+ \omega_N N \int_0^s t^{N-1} \left(\varphi(s-t) - \varphi(s+r-t) \right)^p dt \leq \\ &\leq \omega_N r^p L^p (r+s)^N \end{aligned}$$

where L is the Lipschitz constant of $\varphi(t)$ for small t , and we see that this motion is Lipschitz. If we try to compute

$$\frac{v_{A_t} - v_A}{\|v_{A_t} - v_A\|_{L^p}}$$

we notice that $v_{A_t} - v_A = 0$ outside of B_t : so the limit would be zero for $x \neq 0$.

3.6 Riemannian metric

Let now $p = 2$. The set N_c may fail to be a smooth submanifold of L^2 ; yet we will, as much as possible, pretend that it is, in order to induce a sort of “Riemannian metric” on N_c from the standard L^2 metric.

We define the “Riemannian metric” on N_c simply by

$$\langle h, k \rangle \stackrel{\text{def}}{=} \langle h, k \rangle_{L^2}$$

for $h, k \in T_v N_c$ and correspondingly a norm by

$$|h| \stackrel{\text{def}}{=} \sqrt{\langle h, h \rangle}$$

Proposition 3.22. *We will also argue that the distance induced by this “Riemannian metric” coincides with the geodesically induced distance d^g . Indeed let $\gamma : [a, b] \rightarrow M$ be a Lipschitz curve in N_c ; we may define the “Riemannian length” of the curve*

$$\text{len}^R \gamma \stackrel{\text{def}}{=} \int |\dot{\gamma}| ds$$

Then we define the “Riemannian distance” $d^R(x, y)$ as the infimum of $\text{len}^R \gamma$ for all γ connecting x to y . But by eqn. (14), $\text{len}^R \gamma = \text{len} \gamma$ and $d^R = d^g$.

3.7 Example: smooth convex sets

We propose, as an example, an explicit computation of the Riemannian Metric. We fix $p = 2, N = 2$. Let $\Omega \subset \mathbb{R}^2$ be a convex set with smooth boundary; let $y(\theta) : [0, L] \rightarrow \partial\Omega$ be a parametrization of the boundary, $\nu(\theta)$ the unit vector normal to $\partial\Omega$ and pointing external to Ω : then the following “polar” change of coordinates holds:

$$\psi : \mathbb{R}^+ \times [0, L] \rightarrow \mathbb{R} \setminus \Omega \quad , \quad \psi(\rho, \theta) = y(\theta) + \rho \nu(\theta)$$

We suppose that $y(\theta)$ moves on $\partial\Omega$ in anticlockwise direction; so

$$\nu = J\partial_s y \quad , \quad \partial_{ss}y = -\kappa\nu \quad , \quad \partial_s\nu = \kappa\partial_s y$$

where J is the rotation matrix (of angle $-\pi/2$), κ is the curvature, and $\partial_s y$ is the tangent vector (obtained by deriving y with respect to arc parameter).

We can then express a generic integral through this change of coordinates as

$$\int_{\mathbb{R}^2 \setminus \Omega} f(x) dx = \int_{\mathbb{R}^+} \int_{\partial\Omega} f(\psi(\rho, s)) |1 + \rho\kappa(s)| d\rho ds$$

where s is arc parameter, and ds is integration in arc parameter.

We want to study a smooth deformation of Ω , that we call Ω_t ; then the border $y(\theta, t)$ depends on a time parameter t . Suppose also that $\kappa(\theta) > 0$, that is, that the set is strictly convex: then for small smooth deformations, the set Ω_t will still be strictly convex. By deriving

$$\partial_t \partial_s y = \partial_s (\partial_t y) - \partial_s y \langle \partial_s y, \partial_s (\partial_t y) \rangle = \pi_\nu (\partial_s (\partial_t y))$$

where

$$\pi_\nu(w) \stackrel{\text{def}}{=} w - \nu \langle \nu, w \rangle$$

is the projection of w parallel to ν . Supposing now that $\rho = \rho(t)$ as well, we can express the point $\psi(\rho, y)$ in a first order approximation as

$$d\psi = \left((\partial_t y) + \rho' \nu + \rho J \pi_\nu (\partial_s (\partial_t y)) \right) dt + \left(\partial_\theta y + \rho \partial_\theta \nu \right) d\theta$$

where moreover

$$\left(\partial_\theta y + \rho \partial_\theta \nu \right) d\theta = \left(\partial_s y + \rho \partial_s \nu \right) ds = \left(1 + \rho\kappa \right) \partial_s y ds .$$

If $y(\theta, t)$, $\rho(t)$ are expressing a constant point $x = \psi(\rho, y)$, then $d\psi = 0$; we apply scalar products w.r.t. ν and $\partial_s y$ to the above relations

$$\langle \nu, (\partial_t y) \rangle + \rho' = 0 \quad , \quad \langle \partial_s y, (\partial_t y) \rangle - \rho \langle \nu, \partial_s (\partial_t y) \rangle dt + (1 + \rho\kappa) ds = 0 .$$

Assuming that $(\partial_t y) \perp \partial_s y$, that is, $(\partial_t y) = \alpha \nu$ with $\alpha = \alpha(t, \theta) \in \mathbb{R}$, we obtain the relationships

$$\rho' = -\alpha \quad , \quad \frac{ds}{dt} = \frac{\rho \langle \nu, \partial_s (\alpha \nu) \rangle}{(1 + \rho\kappa)} = \frac{\rho \partial_s \alpha}{(1 + \rho\kappa)} .$$

Now, for $x \notin \Omega_t$, $u_{\Omega_t}(x) = \rho(t)$ hence

$$h_\alpha \stackrel{\text{def}}{=} \partial_t v_{\Omega_t}(x) = -\varphi'(u_{\Omega_t}(x)) \alpha$$

whereas $h_\alpha(x) = 0$ for $x \in \mathring{\Omega}_t$; so h_α is the vector in $T_v N_c$ that is associated to α .

Let us then fix two orthogonal smooth vector fields $\alpha(s)\nu(s)$, $\beta(s)\nu(s)$, that represent two possible deformations of $\partial\Omega$; those correspond to two vectors $h_\alpha, h_\beta \in T_v N_c$; so the Riemannian Metric that we presented in Sec. 3.6 can be pulled back on $\partial\Omega$, to provide the metric

$$\begin{aligned} \langle \alpha, \beta \rangle &\stackrel{\text{def}}{=} \int_{\mathbb{R}^2} h_\alpha(x) h_\beta(x) dx = \int_{\mathbb{R}^2 \setminus \Omega} h_\alpha(x) h_\beta(x) dx = \\ &= \int_{\partial\Omega} \left[\int_{\mathbb{R}^+} (\varphi'(\rho))^2 (1 + \rho\kappa(s)) d\rho \right] \alpha(s) \beta(s) ds \end{aligned}$$

that is,

$$\langle \alpha, \beta \rangle \stackrel{\text{def}}{=} \int_{\partial\Omega} (a + b\kappa(s)) \alpha(s) \beta(s) ds \quad (39)$$

with

$$a = \int_{\mathbb{R}^+} (\varphi'(\rho))^2 d\rho \quad , \quad b = \int_{\mathbb{R}^+} (\varphi'(\rho))^2 \rho d\rho .$$

Smooth sets If Ω is smooth but not convex, then the above formula holds up to the cutlocus. We define a function $R(s) : [0, L] \rightarrow \mathbb{R}^+$ that spans the cutlocus, that is,

$$\text{Cut} = \{\psi(R(s), s), s \in [0, L]\} .$$

ψ is a diffeomorphism between the sets

$$\{(\rho, s) \mid s \in [0, L], 0 < \rho < R(s)\} \leftrightarrow \mathbb{R}^2 \setminus (\Omega \cup \text{Cut})$$

moreover $R(s)$ is Lipschitz (by results in [10],[12]).

In this case the metric has the form

$$\langle h, k \rangle = \int_{\partial\Omega} \left[\int_0^{T(s)} (\varphi'(\rho))^2 (1 + \rho\kappa(s)) d\rho \right] \alpha(s) \beta(s) ds$$

4 Other Banach-like metrics of shapes

The paradigm that we presented in the previous section may be exploited in other similar ways; to conclude the paper, we shortly present some different embeddings (leaving to a future paper the detailed study of their properties).

4.1 Signed distance based representation

We may use the signed distance function b_A , that was defined in (2), to define a metric of shapes:

$$d'(A, B) \stackrel{\text{def}}{=} \|\varphi(b_A) - \varphi(b_B)\|_{L^p(\mathbb{R}^N)}$$

in this case, we require that the function $\varphi : \mathbb{R} \rightarrow (0, \infty)$ is monotonically decreasing and of class C^1 , and such that

$$\varphi(|x| - t) \in L^p(\mathbb{R}^N) \quad \forall t. \quad (40)$$

The resulting metric is slightly stronger than the one we studied in the preceding sections; in particular,

Remark 4.1. Let \mathcal{F} be the class of all finite subsets of \mathbb{R}^N ; this class is dense in \mathcal{I} when we use the metric $d_{p,\varphi}$, or the Hausdorff metric; but it is not dense when we use the metric d' .

4.2 $W^{1,p}$ metrics

Another interesting choice of metric is obtained by embedding the representation in $W^{1,p}$, for $p \in (1, \infty)$

We require that $\varphi : [0, \infty) \rightarrow (0, \infty)$ be Lipschitz, C^1 and monotonically decreasing, and $\varphi(|x|) \in W^{1,p}(\mathbb{R}^N)$; for the case $p < \infty$ we are equivalently asking that

$$\int_0^\infty t^{N-1}(\varphi(t)^p + |\varphi'(t)|^p) dt < \infty$$

and this implies that $\lim_{t \rightarrow \infty} \varphi(t) = 0 = \lim_{t \rightarrow \infty} \varphi'(t)$.

We add one last hypothesis, we assume that there is a $T > 0$ s.t. $\varphi(t)$ is convex for $t \in [T, \infty]$.

Proposition 4.2. *For any A compact we have $v_A \in W^{1,p}(\mathbb{R}^N)$.*

Proof. We already know by 3.2 that $v_A \in L^p(\mathbb{R}^N)$.

By hypotheses above, v_A is Lipschitz; and then, for almost all x , $\nabla v_A = \varphi'(u_A) \nabla u_A$; where $|\nabla u_A| = 1$ for almost all $x \notin A$, while $\nabla u_A = 0$ for almost all $x \in A$. We also know that when $t > T$, $\varphi'(t) < 0$, φ' is increasing and $\varphi'(t) \uparrow 0$.

Let $R > 0$ be large so that $A \subset B_R$, then

$$u_A(x) \geq |x| - R$$

and then when $|x| \geq R + T$ we obtain that

$$\varphi'(u_A(x)) \geq \varphi'(|x| - R)$$

that is

$$\int_{\mathbb{R}^N \setminus B_{R+T}} |\varphi'(u_A(x))|^p dx \leq \int_{\mathbb{R}^N \setminus B_{R+T}} |\varphi'(|x| - R)|^p dx < \infty.$$

At the same time, since v_A is Lipschitz, then $\int_{B_{R+T}} |\nabla v_A| dx$ is finite. \square

Definition 4.3. Given $A, B \in \mathcal{I}$, we define

$$d_{1,p,\varphi}(A, B) \stackrel{\text{def}}{=} \|\varphi(u_A) - \varphi(u_B)\|_{W^{1,p}(\mathbb{R}^N)}$$

We just state a simple property of this metric:

Proposition 4.4. *Let again \mathcal{F} be the class of all finite subsets of \mathbb{R}^N : this class is dense in \mathcal{I} if and only if $\varphi'(0) = 0$.*

Indeed, fix A compact; let $\{x_n\}_n$ be a dense subset of A ; let $A_k \stackrel{\text{def}}{=} \{x_k \mid k \leq n\}$ be a finite subfamily; if $\varphi'(0) = 0$ then $A_k \rightarrow A$ according to $d_{1,p,\varphi}$.

If $\varphi'(0) < 0$ it is easy to find examples where this does not hold: let $N = 1$, $A = [0, 1]$, then $\int_0^1 |v'_{A_k}(t)|^p dt \rightarrow |\varphi'(0)|^p$.

Conclusions

We have studied a metric space of shapes $(\mathcal{I}, d_{p,\varphi})$; this space has a “weak distance”, in that it has many compact sets, and geodesics do exist; but it can be associated in some cases to a smooth Riemannian metric, as we saw in eqn. (39). Moreover, by the properties that we saw in sec. 2.2 (and in particular, by the properties of L^p spaces for $p \in (1, \infty)$ that we proved in Thm. 2.11) we can also hope that geodesics can be studied in the O.D.E. sense (although possibly in a very weak sense).

As we saw in the last chapter, the representation/embedding paradigm can be exploited in many different fashions; we just conclude with one last remark.

Remark 4.5. The embedding of $\varphi \circ u_A$ in $W^{2,p}$ is not feasible: if A is smooth but is not convex, the second derivative of u_A along the cutlocus is expressed by a measure (see 4.13 in [13]) and then $\varphi \circ u_A \notin W^{2,p}$.

Contents

1	Introduction	1
1.1	Shape spaces	2
1.2	Goals	3
1.2.1	Tradeoff	4
1.3	Plan of the paper	5
2	Metric spaces and embeddings in Banach spaces	6
2.1	Metric spaces	6
2.1.1	Distances, quotients and groups	8
2.2	Embeddings in Banach spaces	8
2.2.1	Radon-Nikodym property	9
2.2.2	Embeddings in uniformly convex Banach spaces	9
2.3	Definitions	11
2.4	Hausdorff distance	11
3	L^p-like metrics of shapes	15
3.1	Completeness and compactness	19
3.2	Shape analysis	19
3.3	d^g and geodesics	20
3.4	Variational description of geodesics	22
3.5	Tangent bundle	23
3.6	Riemannian metric	24
3.7	Example: smooth convex sets	24
4	Other Banach-like metrics of shapes	26
4.1	Signed distance based representation	26
4.2	$W^{1,p}$ metrics	26

References

- [1] L. Ambrosio and B. Kirchheim. Rectifiable sets in metric and Banach spaces. *Math. Ann.*, 318:527–555, 2000.
- [2] L. Ambrosio and P. Tilli. *Selected topics in "analysis in metric spaces"*. appunti. edizioni Scuola Normale Superiore, Pisa, 2000.
- [3] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. *Gradient flows in metric spaces and in the space of probability measures*. Birkhäuser, 2004.
- [4] H. Brezis. *Analisi Funzionale*. Liguori Editore, Napoli, 1986. (italian translation of *Analyse fonctionnelle*, Masson, 1983, Paris).
- [5] G. Charpiat, O. Faugeras, and R. Keriven. Approximations of shape metrics and application to shape warping and empirical shape statistics. INRIA report 4820, 2003.

- [6] M.C. Delfour and J.P. Zolésio. *Shape and Geometries*. Advances in Design and Control. SIAM, 2001.
- [7] Alessandro Duci, Anthony J. Yezzi, Sanjoy K. Mitter, and Stefano Soatto. Shape representation via harmonic embedding. In *International Conference on Computer Vision (ICCV03)*, volume 1, pages 656 – 662, Washington, DC, USA, 2003. IEEE Computer Society.
- [8] Alessandro Duci, Anthony J. Yezzi, Stefano Soatto, and Kelvin Rocha. Harmonic embeddings for linear shape. *J. Math Imaging Vis*, 25:341–352, 2006.
- [9] Joan Glaunès, Alain Trounev, and Laurent Younes. Modeling planar shape variation via Hamiltonian flows of curves. In Anthony Yezzi and Hamid Krim, editors, *Analysis and Statistics of Shapes*, Modeling and Simulation in Science, Engineering and Technology, chapter 14. Birkhäuser - Springer - Verlag, 2005.
- [10] J. Itoh and M. Tanaka. The Lipschitz continuity of the distance function to the cut locus. *Trans. A.M.S.*, 353(1):21–40, 2000.
- [11] Eric Klassen, Anuj Srivastava, Washington Mio, and Shantanu Joshi. Analysis of planar shapes using geodesic paths on shape spaces. 2003.
- [12] YanYan Li and Louis Nirenberg. The distance function to the boundary, Finsler geometry and the singular set of viscosity solutions of some Hamilton-Jacobi equations. *Comm. Pure Appl. Math.*, LVIII, 2005. (first received as a personal communication in June 2003).
- [13] C. Mantegazza and A. C. Mennucci. Hamilton–Jacobi equations and distance functions on Riemannian manifolds. *Applied Math. and Optim.*, 47(1):1–25, 2002.
- [14] A. C. G. Mennucci. On asymmetric distances. preprint, <http://cvgmt.sns.it/papers/and04/>.
- [15] Andrea Mennucci, Anthony Yezzi, and Ganesh Sundaramoorthi. Properties of sobolev active contours. arxiv:math.DG.0605017; submitted to I.F.B., 2006.
- [16] Peter W. Michor and David Mumford. Vanishing geodesic distance on spaces of submanifolds and diffeomorphisms. *Documenta Math.*, 10:217–245, 2005.
- [17] Peter W. Michor and David Mumford. An overview of the riemannian metrics on spaces of curves using the hamiltonian approach. 2006.
- [18] Peter W. Michor and David Mumford. Riemannian geometris of space of plane curves. *J. Eur. Math. Soc. (JEMS)*, 8:1–48, 2006.
- [19] Washington Mio and Anuj Srivastava. Elastic-string models for representation and analysis of planar shapes. In *Conference on Computer Vision and Pattern Recognition (CVPR)*, June 2004.
- [20] Anuj Srivastava, Shantanu Joshi, Washington Mio, and Xiuwen Liu. Statistical shape analysis: Clustering, learning, and testing. 2004.
- [21] Ganesh Sundaramoorthi, Anthony Yezzi, and Andrea Mennucci. Sobolev active contours. In *VLSM 2005*, 2005. <http://vlsm05.enpc.fr/programme.htm>.

- [22] Ganesh Sundaramoorthi, Anthony Yezzi, and Andrea Mennucci. Sobolev active contours. Technical report, GaTech, 2005. <http://users.ece.gatech.edu/~ganeshs/sobolev/sobolev.html> , <http://users.ece.gatech.edu/~ganeshs/sobolev/pubs/techrep.pdf>.
- [23] Ganesh Sundaramoorthi, Anthony Yezzi, and Andrea Mennucci. Tracking with Sobolev active contours. In *Conference on Computer Vision and Pattern Recognition (CVPR06)*. IEEE Computer Society, 2006.
- [24] Ganesh Sundaramoorthi, Anthony Yezzi, and Andrea Mennucci. Coarse-to-fine segmentation and tracking using Sobolev active contours. *IEEE Transactions on Pattern Analysis and Machine Intelligence (TPAMI)*, 2007.
- [25] Ganesh Sundaramoorthi, Anthony Yezzi, and Andrea Mennucci. Sobolev active contours. *Intn. Journ. Computer Vision*, 2007. <http://dx.doi.org/10.1007/s11263-006-0635-2>.
- [26] Ganesh Sundaramoorthi, Anthony Yezzi, Andrea Mennucci, and Guillermo Sapiro. New possibilities with Sobolev active contours. In *Scale Space Variational Methods 07*, 2007. "Best Numerical Paper-Project Award" , http://ssvm07.ciram.unibo.it/ssvm07_public/index.html.
- [27] Alain Trounev and Laurent Younes. Local geometry of deformable templates. *SIAM J. Math. Anal.*, 37(1):17–59 (electronic), 2005.
- [28] A. Yezzi and A. Mennucci. Geodesic homotopies. In *EUSIPCO04*, 2004.
- [29] A. Yezzi and A. Mennucci. Metrics in the space of curves. *arXiv*, 2004. [arXiv:math.DG/0412454](http://arxiv.org/abs/math.DG/0412454).
- [30] A. Yezzi and A. Mennucci. Conformal metrics and true “gradient flows” for curves. In *International Conference on Computer Vision (ICCV05)*, 2005. <http://research.microsoft.com/iccv2005/>.
- [31] Laurent Younes. Computable elastic distances between shapes. *SIAM Journal of Applied Mathematics*, 58:565–586, 1998.